

Some Features of Scattering Problem in a κ -Deformed Minkowski Spacetime

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Abstract

The doubly special relativity (DSR) theories are suggested in order to incorporate an observer-independent length scale in special theory of relativity. The Magueijo-Smolín proposal of DSR is realizable through a particular form of the noncommutative (NC) spacetime (known as κ -Minkowski spacetime) in which the Lorentz symmetry is preserved. In this framework, the NC parameter κ provides the origin of natural cutoff energy scale. Using a nonlinear deformed relativistic dispersion relation along with the Lorentz transformations, we investigate some phenomenological facets of two-body collision problem (without creation of new particles) in a κ -Minkowski spacetime. By treating an elastic scattering problem, we study effects of the Planck scale energy cutoff on some relativistic kinematical properties of this scattering problem. The results are challenging in the sense that as soon as one turns on the κ -spacetime extension, the nature of the two-body collision alters from elastic to inelastic one. It is shown also that a significant kinematical variable involving in heavy ion collisions, the rapidity, is not essentially an additive quantity under a sequence of the nonlinear representation of the Lorentz transformations.

Keywords: Doubly Special Relativity; Noncommutative Spacetime; Nonlinear Lorentz Transformations; Elastic Scattering; Rapidity.

1 Introduction

Lorentz symmetry (LS), from both theoretical and experimental perspectives, has a very special status in modern physics so that it has attracted much attention these years. More than half a century, we are faced with a wide variety of researches that are focused on verifying the authenticity of the LS. A notable number of these studies were able to show that in quantum gravity (Planck scale) regime, there is the possibility of violation of LS, see for instance [1]-[17]. This issue reflects the fact that LS is not necessarily an exact symmetry of the nature in all energy scales. Rather, it seems to be only an approximate symmetry governed on low energy scales, so that in the Planck energy scale it loses credibility due to existence of a *preferred reference frame*. Indeed, the existence of a preferred state of motion leads to a gross violation of the principle of relativity (that the laws of physics are the same for all inertial observers) at the Planck energy scale. In other words, in Planck scale there may be a reference frame in which the laws of physics might appear to be different from those in other frames. From a cosmological viewpoint, it has an explicit consequence that there may be a preferred cosmological rest frame [18]. However, along with violation of Lorentz invariance around Planck scale, further conceptual challenges such as the invalidity of the equivalence principle at that energy scale arise automatically. A common feature of different theories trying to provide a coherent description of Planck scale physics is that the geometric structure of spacetime at this scale is noncommutative

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(NC) [19, 20]. In other words, in Planck scale the NC field theory is governed as a framework of spacetime discreteness [21]. In fact NC geometry is a powerful mathematical framework to describe a natural quantization of manifolds [22]. Historically, this issue for the first time was sparked by the Gelfand-Naimark theorem [23]. Technically, this theorem expresses the fact that there is a one to one correspondence between specific commutative algebras and specific spaces i.e. a duality between commutative C^* algebras and locally compact spaces. Indeed, this theorem indicates that geometric structure of certain spacetimes can be seen within certain algebraic representations. So, extension of commutative algebras to NC ones provides the NC algebraic structure on NC spacetimes. Since the standard Minkowski spacetime at the quantum gravity level becomes quantized, it seems to be natural that the commutative algebra of coordinates x^μ on four dimensional real vector space (i.e. $[x^\mu, x^\nu] = 0$) are replaced by NC algebra $[x^\mu, x^\nu] \neq 0$. Originally the NC algebraic relations were formally introduced as $[x^\mu, x^\nu] = i\theta^{\mu\nu}$ where $\theta^{\mu\nu}$ denotes a constant C -number. These algebraic relations found considerable popularity in quantum field theory (QFT) and also string theory. For instance, the seminal work of Seiberg and Witten [24] can be mentioned. They showed that in certain low energy limit of open strings traveling in the background of a two form gauge field, the NC manifold arises naturally. The important thing to note is that QFTs defined on the present form of NC spacetime, do not meet the LS. Latter on, the following Lie-algebraic form of NC algebra with structure constants $\theta_\lambda^{\mu\nu}$ have been introduced [25]-[33]

$$[x^\mu, x^\nu] = i\theta_\lambda^{\mu\nu} x^\lambda. \quad (1)$$

In fact, κ -Minkowski space or algebra is a restricted class of this algebra which obeys the following commutation relations

$$[x^0, x^i] = \frac{i}{\kappa} x^i, \quad [x^0, x^0] = [x^i, x^j] = 0, \quad i, j = 1, 2, 3 \quad (2)$$

Here, x^0 and x^i signify the time and space operators, respectively. For a detailed study on the mathematical formalism of κ -Minkowski spacetimes along with some of its applications in Planck scale physics, see [34]-[37]. Many endeavors have been done to make a QFT relied strongly on a κ -Minkowski spacetime. For instance one can mention [38]-[47]. Unlike prior studies, in the present context one is not dealing with violation of LS. On the other hand, Amelino-Camelia [48] along with Magueijo and Smolin (MS) [49], independently proposed alternative scenarios to special relativity (SR) which is called “*doubly special relativity*” (DSR). In these scenarios there is an extra invariant, *Planck length or energy* apart from *the speed of light*. Note that both of these scenarios address a type of QG with no imperil of the LS due to respecting *relativity of inertial frames*. Technically, in DSRs there is not necessarily a breaking of LS, rather it contains just *deformation* of this symmetry. Interestingly, in Ref. [50] it has been shown that κ -Minkowski space is a realization of the DSR, to the extent that thought to be one of the most prosperous possibilities of DSR proposal, at least so far. More precisely, using the co-product of κ -Poincaré algebra¹ In other words, the spacetime structure of the DSR proposal is equivalent to NC spacetime one so that it suggests a NC version of Minkowski spacetime satisfying the LS. Therefore, DSR as an effective approach to QG, expressly predicts an energy cutoff κ equivalent to an observer independent threshold length scale. It is striking that from existence of such a threshold length scale one can justify the lack of spontaneous formation of black holes in a very condense district of spacetime [51].

By approaching the length scales comparable to l_p , since continuous metric idea fails, one expects quantities such as the relativistic mass-shell condition (or dispersion relation) generally gets modified to have the form such as

$$E^2 = p^2 + m^2 + l_p E^3 + \dots. \quad (3)$$

¹Note that as authors of Ref. [50] have shown, there are infinitely many DSR constructions of the energy-momentum and each of these constructions can be promoted to the κ -Poincaré quantum algebra. along with the construction of κ -deformed phase space, the NC spacetime structure can be extracted as well as the entire DSR phase space.

This general form of the modified dispersion relation was suggested initially by Amelino-Camelia and Piran [52]. Scenarios with modified dispersion relations can be used in diverse areas of physics; to learn more about some of these applications see for instance [53]-[60]. It is important to note that in the context of DSR, addition of some *nonlinear* terms to the Lorentz transformations foils appearance of paradox between the existence of an observer independent threshold length scale from one side and lack of Lorentz invariance of length in standard SR theory from other side, see [48, 49] and also [61, 62, 63, 64] for a technical review. As a consequence of nonlinear extension of the ordinary Lorentz group, one can say that in DSR framework spacetime background coordinates are NC in essence. Therefore, to analyze the spacetime symmetries one should refer to quantum groups (see [65]-[68] for a review of various aspects of DSRs in this respect). In the present work, we focus mainly on the Magueijo-Smolín DSR proposal with assumption that the geometry of Minkowski spacetime background deforms through the introduction of a NC geometry parameter κ . As a result, the relativistic mass-shell condition can be modified as follows

$$E^2 - p^2 c^2 = m^2 c^4 \left(1 - \frac{E}{\kappa}\right)^2, \quad (4)$$

where E and p represent the magnitude of the energy and the three-momentum of the particle with rest mass m , respectively². It has been shown that to preserve the form of the modified mass-shell condition in all inertial frames, the usual law of energy-momentum conservation should be corrected too [48]. Regarding the κ -Minkowski spacetime as made with NC coordinates x^α that fulfill the Lie-algebra of the type (2), then Jacobi identity does not allow the canonical commutation relations $\{x_\mu, p_\nu\} = -g_{\mu\nu}$ to be unchanged. Therefore, Poisson brackets between the coordinates of the phase space in the κ -Minkowski spacetime obey the following modified algebra [71]-[74]

$$\begin{aligned} \{x_\mu, x_\nu\} &= \frac{1}{\kappa}(x_\mu \theta_\nu - x_\nu \theta_\mu), \\ \{x_\mu, p_\nu\} &= -g_{\mu\nu} + \frac{1}{\kappa} \eta_{\mu\nu} p_\nu, \\ \{p_\mu, p_\nu\} &= 0, \end{aligned} \quad (5)$$

where $\mu, \nu = 0, 1, 2, 3$ and $\theta_0 = 1, \theta_{1,2,3} = 0$. It is straightforward to check that in the limit of $\kappa \rightarrow \infty$, this algebra reduces to the usual Minkowski Lie algebra with the conventional canonical commutator relations between coordinates of the phase space. So far, we have learned that the laws of DSR are nonlinear extension of the standard SR theory so that the variables of the DSR are connected to its SR counterparts through a nonlinear mapping [75, 76]. In this respect, some works have been developed in order to study mapping between κ -Minkowski spacetime and its Minkowski counterpart through realization formalism, see for instance [77]-[81]. We emphasize that for NC spaces there exist generally infinitely many realizations in terms of commutative coordinates. Nevertheless, the physical outputs should be independent of these realizations [81]. Note also that due to nontrivial change of the Poisson brackets (5), the mentioned map will be non-canonical. In fact, the nonlinear mapping between DSR and SR never results in one to one correspondence between these two scenarios. Rather, from a phenomenological perspective, DSR can provide a new extended framework independent of SR one. In this respect we would stress that treatments which describe DSRs in terms of the maps

²To give a quantitative estimation of the energy scale needed to observe the expected experimental deviations from relativistic mass-shell relation, one may be faced with the reasonable phenomenological question that whether the experimental predictions always are confined to the unapproachable Planckian regime. Naturally deformed mass-shell condition and subsequent experimental deviations should be valid and traceable at the " κ -scale" i.e. QG scale. But, really at what scale QG effects become important? A general answer to this question is that QG effects must be governed at scale $l_{QG} = 1/M_{QG}$ (here $\hbar = 1 = c$) so that M_{QG} is expected to be on the order of the Planck mass, M_{Planck} . It is more than a decade that serious attempt have began to bounding M_{QG} by measuring quantum gravitational effects through dispersion relations of high energetic photons released from astrophysical sources such as gamma ray bursts that can be detected by apparatus such as the Fermi telescope [69]. An explicit bound reported in these studies is $M_{QG} > 0.1 M_{Planck}$; for details see the discussion raised by Amelino-Camelia and Smolin in [70].

from SR are usually misleading. For instance some authors (e.g. [82]) based on such a treatment have claimed that DSRs [48, 49] are not a new relativity. Amelino-Camelia in [83] explicitly has responded to such a misconception.

Without getting into technical details, in what follows we just introduce nonlinear DSR Lorentz transformations for our future purposes in this work (one can refer to [75, 76] for a detailed discussion). If we restrict the boost to the x^1 direction with velocity $u^{1,2,3} = (u, 0, 0)$, then for 4-vectors $x^\mu = (t, \frac{x^{1,2,3}}{c})$ and $p^\mu = (\frac{E}{c}, p^{1,2,3})$, modified Lorentz transformations arising from NC geometry parameter κ (as a realization of Magueijo-Smolín proposal of DSR) are written as

$$t' = \gamma\alpha(t - \frac{u}{c^2}x^1), \quad x'^1 = \gamma\alpha(x^1 - \frac{u}{c^2}x^0), \quad x'^2 = \alpha x^2, \quad x'^3 = \alpha x^3. \quad (6)$$

and

$$E' = \frac{\gamma}{\alpha}(E - \frac{u}{c^2}p^1), \quad p'^1 = \frac{\gamma}{\alpha}(p^1 - \frac{u}{c^2}p^0), \quad p'^2 = \frac{p^2}{\alpha}, \quad p'^3 = \frac{p^3}{\alpha}. \quad (7)$$

respectively, where $\gamma = (1 - \frac{u^2}{c^2})^{-1/2}$ is the dimensionless Lorentz contraction factor and

$$\alpha = 1 + \frac{[(\gamma - 1)p^0 - \gamma u p^1]}{\kappa}.$$

It is obvious that in the limit of $\kappa \rightarrow \infty$ (i.e. in the absence of an upper bound for energy scale), $\alpha \rightarrow 1$ and the above transformations reduce to the standard relativistic transformations. These transformations show that the effects of NC geometry parameter is traceable via nonlinear transformation rules [84].

Despite the fact that DSRs suffer from the lack of a consistent and well-established mathematical structure similar to what exists for SR, κ -Minkowski spacetime and κ -Poincaré are likely the richest frameworks which can be assigned to DSRs, at least so far. Therefore, in the present paper, we just adopt the DSR interpretation proposed in Magueijo-Smolín model [49] as one of the conceivable phenomenological frameworks of κ -Minkowski spacetime. As an interesting and promising feature of the mentioned framework, it can be used to explain astrophysical data received from GRBs (note that Gamma-ray bursts (GBRs) along with supernovae, neutron stars and black holes are four main pillars of relativistic astrophysics) [85].

With these preliminaries and in the context of some prevalent phenomenological issues through combination of the modified mass shell condition (4) and κ -deformed Lorentz transformations (6) and (7), we examine and derive some phenomenological consequences of a κ -Minkowski relativistic model. Specifically, we treat the elastic scattering problem with the Planck energy cutoff by focusing on some relativistic kinematical properties of this scattering problem. We are looking for the mentioned objectives in these steps: In section 2 we study the effects of the Planck energy upper bound, κ , on the elastic scattering process where no new particles are created during the process. Section 3 is devoted to the investigation of the *rapidity* as one of the most important kinematical variables involving in the heavy ion collisions in the presence of the κ deformation of the Minkowski spacetime. The paper follows with a conclusion along with a brief remark in section 4.

2 κ -Deformed Elastic Scattering

In this section, we investigate the elastic scattering processes where no new particles are created during the process in a κ -deformed Minkowski space-time. To this end, we firstly present an explicit relation for κ -deformed energy and momentum. Suppose a clock that is fixed at the position x_1 of a rest frame, S . If the clock emits signals in a regular time interval $\Delta t = t_2 - t_1$, then according to the modified Lorentz transformations (6), an observer located in the moving system S' , measures this time interval as follows

$$\Delta t' = \gamma\alpha \left[\left(t_2 - \frac{u}{c^2}x_2 \right) - \left(t_1 - \frac{u}{c^2}x_1 \right) \right]. \quad (8)$$

Since the clock is fixed at the system S , then $x_1 = x_2$, which gives

$$\Delta t' = \alpha \gamma \Delta t . \quad (9)$$

This equation can be rewritten in terms of the proper time τ as

$$\Delta t' = \alpha \gamma \Delta \tau . \quad (10)$$

In this framework we have

$$p = m \frac{d\mathbf{x}}{d\tau} = m \frac{d\mathbf{x}}{dt} \cdot \frac{dt}{d\tau} = \alpha \gamma m \mathbf{u} , \quad (11)$$

where $\mathbf{u} = \frac{d\mathbf{x}}{dt}$ is the classical velocity and m marks the rest mass. Now, inserting the κ -deformed relativistic momentum (11) into the modified dispersion relation (4), we derive the following equation

$$\left(1 - \frac{m^2 c^4}{\kappa^2}\right) E^2 + \frac{2m^2 c^4}{\kappa} E - m^2 c^4 [1 + \alpha^2 (\gamma^2 - 1)] = 0 . \quad (12)$$

Solving this equation we find

$$E_\kappa = \frac{-\frac{2m^2 c^4}{\kappa} \pm \left[\frac{4m^4 c^8 \alpha^2 (\gamma^2 - 1)}{\kappa^2} + 4m^2 c^4 [1 + \alpha^2 (\gamma^2 - 1)] \right]^{1/2}}{2 \left(1 - \frac{m^2 c^4}{\kappa^2}\right)} . \quad (13)$$

The positive sign in this solution is acceptable since for the case with $\kappa \rightarrow \infty$ and $\alpha \rightarrow 1$, naturally the κ -deformed energy (13) should reduce to the relativistic result $E = \gamma m c^2$. By applying the approximation $\frac{m^2 c^4}{\kappa^2} \ll 1$, Eq. (13) can be rewritten as

$$E_\kappa = \frac{m^2 c^4}{\kappa} \xi , \quad (14)$$

where

$$\xi = -1 + \sqrt{\alpha^2 (\gamma^2 - 1) + \left(\frac{\kappa}{m c^2}\right)^2 [1 + \alpha^2 (\gamma^2 - 1)]} , \quad (15)$$

is a dimensionless running constant. Note that $\lim_{\kappa \rightarrow \infty} E_\kappa = \gamma m c^2$. In order to study the classical collision process, the center of mass (CM) coordinate system is an appropriate tools to derive many of kinematic relations, while in the relativistic framework it is meaningless to speak about the CM system. In SR theory, mass and energy are related so that it is common in SR kinematics that one uses a center of momentum coordinate system instead of the CM. Of course, in the center of momentum coordinate as CM, the total linear momentum of the system is zero. In the context of the relativistic collision, the laboratory coordinate system is related to the inertial system S and the center of momentum system S' (the moving system) via a Lorentz transformation. In DSR framework we follow the standard procedure with the difference that laboratory system S and the moving system S' are related now by the κ -deformed Lorentz transformations. So, if a particle of rest mass m_1 which moves in one dimension collides elastically with a particle of the rest mass m_2 , then in the center of momentum system, we have³

$$p'_1 = p'_2 . \quad (16)$$

At the first glance this equality seems to be misleading. It is obvious that from the deformed Lorentz transformations one naturally gets the deformed dispersion relation (4) (since this transformation keeps (4) invariant). Therefore, one has also the deformed conservation of energy-momentum so that this relation then governs on the nature of how particles behave in collisions. More precisely, one

³Here $p'_{1,2}$ denote the magnitude of the 3-momentum of particles $m_{1,2}$ in the CM system so that $\mathbf{p}'_1 + \mathbf{p}'_2 = 0$. An observer located in the CM frame sees these two particles are moving towards each other so that depending on the direction, one has $p'_1 - p'_2 = 0$ or $p'_2 - p'_1 = 0$.

expects that in the standard momentum conservation law (16), κ -deformation must be regarded. While this is sensible in essence, as we show equation (16) is still valid up to the first order correction. By using the relation $\beta\gamma = \sqrt{\gamma^2 - 1}$, the space components of the momentum 4-vector in (16) can be written as

$$m_1 c \alpha \sqrt{\gamma_1'^2 - 1} = m_2 c \alpha \sqrt{\gamma_2'^2 - 1}, \quad (17)$$

since $p_1 = \alpha \gamma_1 m_1 u_1$, $p_2 = \alpha \gamma_2 m_2 u_2$ and also $\beta \equiv \frac{u}{c}$. According to the deformed Lorentz transformation (7) and using (14), the transformation of the momentum p_1 (from S to S') is written as

$$p_1' = m_1 c \beta_1 \gamma_1' \gamma_2' - \frac{m_1^2 c^3}{\kappa \alpha} \beta_2' \gamma_2' \xi_1. \quad (18)$$

Finally by replacing this relation into the left hand side of relation (17), we arrive at the following relation

$$m_1 c \beta_1 \gamma_1' \gamma_2' - \frac{m_1^2 c^3}{\kappa \alpha} \beta_2' \gamma_2' \xi_1 = m_2 c \alpha \sqrt{\gamma_2'^2 - 1}, \quad (19)$$

where this relation can be solved for γ_1' and γ_2' in terms of γ_1 to find

$$\gamma_1' = \frac{\left(\frac{m_1}{m_2} \alpha + \frac{m_1 c^2}{\kappa \alpha} \xi_1 \right)}{\sqrt{(1 - \gamma_1^2) + \left(\frac{m_1}{m_2} \alpha + \frac{m_1 c^2}{\kappa \alpha} \xi_1 \right)^2}}, \quad (20)$$

and

$$\gamma_2' = \frac{\left(\frac{m_2}{m_1} \alpha + \frac{m_1 c^2}{\kappa \alpha} \xi_1 \right)}{\sqrt{(1 - \gamma_1^2) + \left(\frac{m_2}{m_1} \alpha + \frac{m_1 c^2}{\kappa \alpha} \xi_1 \right)^2}}. \quad (21)$$

In the absence of the κ deformation of the Minkowski space-time background i.e. for $\kappa \rightarrow \infty$, then $\alpha = 1$ and $\frac{m_0 c^2}{\kappa \alpha} \xi_1 = \gamma_1$. So, the above equations reduce to the flowing special relativistic counterparts

$$\gamma_1' = \frac{\left(\frac{m_1}{m_2} + \gamma_1 \right)}{\sqrt{1 + 2\gamma_1 \left(\frac{m_1}{m_2} \right) + \left(\frac{m_1}{m_2} \right)^2}}, \quad (22)$$

and

$$\gamma_2' = \frac{\left(\frac{m_2}{m_1} + \gamma_1 \right)}{\sqrt{1 + 2\gamma_1 \left(\frac{m_2}{m_1} \right) + \left(\frac{m_2}{m_1} \right)^2}}. \quad (23)$$

Now we write the transformation equations for momentum components between the moving system S' and the laboratory system S after the scattering. It is obvious that after scattering we have both x and y components of the momentum. The x and y components of the momentum after scattering in the laboratory system S read as follows

$$p_{1,x} = m_1 c \gamma_2' \left(\beta_1 \alpha \gamma_1' \cos \theta + \frac{m_1 c^2}{\kappa \alpha} \beta_2' \xi_1' \right), \quad (24)$$

and

$$p_{1,y} = m_1 c \beta_1' \gamma_1' \alpha \sin \theta. \quad (25)$$

By introducing the angle of scattering ψ in the laboratory frame and by dividing (25) with (24), we get

$$\tan \psi = \frac{\alpha \sin \theta}{\gamma_2' \left(\alpha \cos \theta + \frac{m_1 c^2}{\kappa \alpha \gamma_1'} \left(\frac{\beta_2'}{\beta_1'} \right) \xi_1' \right)}, \quad (26)$$

where by inserting $p'_1 = p'_2$, it takes the following form

$$\tan \psi = \frac{\alpha \sin \theta}{\gamma'_2 \left(\alpha \cos \theta + \frac{m_1 c^2}{\kappa \alpha \gamma'_1} \left(\frac{m_1 \gamma'_1}{m_2 \gamma'_2} \right) \xi'_1 \right)} . \quad (27)$$

By the same procedure, for the recoiled particle we find respectively

$$p_{2,x} = m_2 c \gamma'^2_2 \beta' \left(\frac{m_2 c^2}{\kappa \alpha \gamma'_2} \xi'_2 - \cos \theta \right) , \quad (28)$$

and

$$p_{2,y} = -m_2 c \beta'_2 \gamma'_2 \alpha \sin \theta . \quad (29)$$

Introducing an angle of scattering η in the laboratory frame for the recoiled particle and then by dividing (29) with (28), we find

$$\tan \eta = - \frac{\alpha \sin \theta}{\gamma'_2 \left(\frac{m_2 c^2}{\kappa \alpha \gamma'_2} \xi'_2 - \cos \theta \right)} . \quad (30)$$

For the special case with $m_1 = m_2$, i.e. with

$$\gamma'_1 = \gamma'_2 = \frac{1}{f(\gamma_1)} , \quad (31)$$

we find

$$f(\gamma_1) = \sqrt{1 + \frac{\alpha^2(1 - \gamma_1^2)}{\left(\alpha^2 + \sqrt{1 + \alpha^2(\gamma_1^2 - 1)} \right)^2}} . \quad (32)$$

Therefore, for the case $m_1 = m_2$, the angles ψ and η take the following forms respectively

$$\psi = \arctan \left(\frac{\alpha \sin \theta}{\frac{\alpha \cos \theta}{f(\gamma_1)} + \sqrt{\frac{1}{f^2(\gamma_1)} + \frac{1}{\alpha^2} - 1}} \right) , \quad (33)$$

and

$$\eta = \arctan \left(\frac{\alpha \sin \theta}{-\frac{\alpha \cos \theta}{f(\gamma_1)} + \sqrt{\frac{1}{f^2(\gamma_1)} + \frac{1}{\alpha^2} - 1}} \right) , \quad (34)$$

So, the angle between the directions of the scattered and recoiled particles is given by $\phi = \psi + \eta$. Through the relation $\frac{d\phi}{d\theta} = 0$, one finds that for $\theta = \pm \frac{\pi}{2}$ the angle ϕ has the maximum

$$\phi_{\kappa,Max} = 2 \arctan \left(\frac{\alpha}{\sqrt{\frac{1}{f^2(\gamma_1)} + \frac{1}{\alpha^2} - 1}} \right) \quad (35)$$

As expected, for the case $\alpha \rightarrow 1$, Eq. (35) recovers its relativistic counterpart as

$$\phi_{SR,Max} = 2 \arctan \sqrt{\frac{2}{1 + \gamma_1}} . \quad (36)$$

Therefore, one finds that in the κ -deformed non-commutative geometry, the maximum amount of the included angle ϕ is dependent on the two dimensionless parameters γ_1 and α . In order to have a qualitative understanding of Eq. (35), we plot variation of ϕ_{Max} in terms of γ_1 in figure 1. If we set the α to be a running constant, this figure gives a qualitative description of the situation. For any given value of γ_1 , by increasing the value of α the maximum included scattering angle ϕ_{Max} increases.

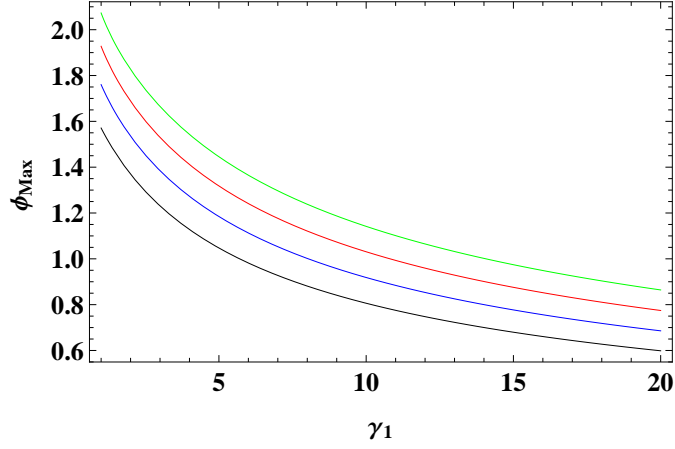


Figure 1: Variation of the maximum included scattering angle ϕ_{Max} in terms of the dimensionless Lorentz contraction factor γ_1 , for different values of α due to κ -deformed Lorentz transformation: $\alpha = 1$ (Black), $\alpha = 1.1$ (Blue), $\alpha = 1.2$ (Red) and $\alpha = 1.3$ (Green). The unit of ϕ is in radian.

So, we can say that correction of α due to the κ -deformed Minkowski spacetime results in a shift of the maximum included scattering angle ϕ_{Max} .

In what follows, to see another prediction of the setup, we assume that the recoil angle η is negligible. Then we track the effect of κ modification on the kinetic energy K of the scattered particle 1 by the target particle 2. Using the inverse κ -modified Lorentz transformation (7), we have

$$E_{1,a} = \frac{\gamma'_1}{\alpha} \left(E'_{1,a} + c\beta'_1 p'_{1,x} \cos \theta \right), \quad (37)$$

where $E_{1,a}$ and $E'_{1,a}$ represent the total energy of particle 1 after collision in the laboratory and center of momentum frames, respectively. By substituting $E'_{1,a} = mc^2 \gamma'_1$ and $p'_{1,x} = mc\beta'_1 \gamma'_1$ into Eq. (37), we get

$$E_{1,a} = mc^2 \frac{\gamma'^2_1}{\alpha} (1 + \beta'^2_1 \cos \theta). \quad (38)$$

By introducing $K_{1,b}$ and $K_{1,a}$ as the kinetic energies of the particle 1 before and after scattering respectively, then one can write

$$\frac{K_{1,a}}{K_{1,b}} = \frac{\frac{\gamma'^2_1}{\alpha} + \frac{\gamma'^2_1 - 1}{\alpha} \cos \theta - 1}{\gamma_1 - 1}, \quad (39)$$

Using equation (31), this equation can be rewritten as follows

$$\frac{K_{1,a}}{K_{1,b}} = \frac{\frac{1}{\alpha f^2(\gamma_1)} + \frac{1}{\alpha} \left(\frac{1}{f^2(\gamma_1)} - 1 \right) \cos \theta - 1}{\gamma_1 - 1}. \quad (40)$$

Now we introduce a scattering angle θ in the center of momentum frame in terms of the ψ in the laboratory frame. For this purpose, by squaring Eq. (33), we find

$$A_1 \cos^2 \theta + A_2 \cos \theta + A_3 = 0, \quad (41)$$

where by definition

$$\begin{aligned} A_1 &\equiv \alpha^2 \left(1 + \frac{\tan^2 \psi}{f^2(\gamma_1)} \right), \\ A_2 &\equiv 2 \tan^2 \psi \sqrt{\frac{\alpha^2}{f^4(\gamma_1)} + \frac{(1-\alpha^2)}{f^2(\gamma_1)}}, \\ A_3 &\equiv \tan^2 \psi \left(\frac{1}{f^2(\gamma_1)} + \frac{1}{\alpha^2} - 1 \right) - \alpha^2. \end{aligned} \quad (42)$$

The solution of equation (41), after some rearrangement, reads as follows

$$\begin{aligned} \cos \theta = & \frac{-\frac{\tan^2 \psi}{f^2(\gamma_1)} \sqrt{\frac{1}{\alpha^2} + \frac{f^2(\gamma_1)(1-\alpha^2)}{\alpha^4}}}{1 + \frac{\tan^2 \psi}{f^2(\gamma_1)}} \\ & \pm \frac{1}{4[\alpha^4 \sin^2 \psi + \alpha^2 f^2(\gamma_1) \cos^2 \psi]} \times \\ & \left\{ \left[4\alpha^6 f^2(\gamma_1) - 8\alpha^8 f^2(\gamma_1) + 4\alpha^4 f^4(\gamma_1) + 4\alpha^8 f^4(\gamma_1) \right] \right. \\ & \cos^2(2\psi) + 8\alpha^8 f^4(\gamma_1) \cos(2\psi) - 4\alpha^6 f^2(\gamma_1) \\ & + 4\alpha^8 f^2(\gamma_1) - 4\alpha^4 f^4(\gamma_1) + 4\alpha^6 f^4(\gamma_1) + \\ & \left. 4\alpha^8 f^4(\gamma_1) \right\}^{1/2} \end{aligned} \quad (43)$$

Through a straightforward calculation, one can show that equation (43) in the limit of $\alpha \rightarrow 1$ recovers its special relativistic counterpart as follows

$$\cos \theta = \frac{-\frac{(\gamma_1+1)}{2} \tan^2 \psi \pm 1}{1 + \frac{(\gamma_1+1)}{2} \tan^2 \psi}. \quad (44)$$

Finally, by inserting this relation along with (32) into equation (40), we are able to show the qualitative behavior of $\frac{K_{1,a}}{K_{1,b}}$ in terms of the laboratory scattering angle ψ , as shown in figure 2. As we see in this

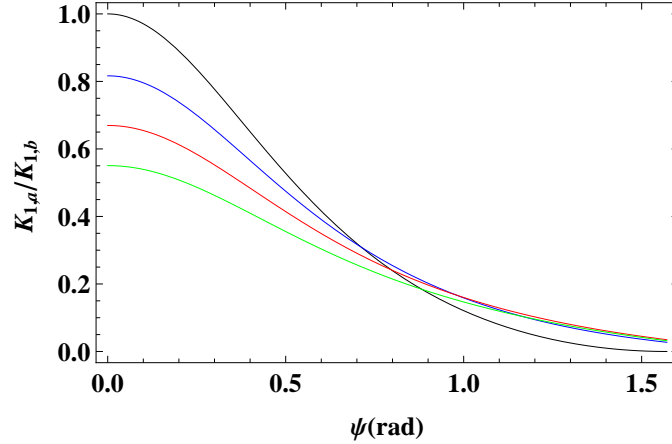


Figure 2: Variation of $\frac{K_{1,a}}{K_{1,b}}$ in terms of Lab scattering angle ψ for different values of α due to κ -deformed Lorentz transformation: $\alpha = 1$ (Black), $\alpha = 1.1$ (Blue), $\alpha = 1.2$ (Red) and $\alpha = 1.3$ (Green). We also set the dimensionless relativistic parameter $\gamma = 10$ which is equivalent to $u \approx 0.99c$. The unit of ψ is in radian.

figure, contrary to our expectation based on the SR theory ($\alpha = 1$) and even classical mechanics⁴ (that is, $\alpha = 1 = \gamma$, in Eqs. (40) and (44)), for the case $\psi = 0$ we have $\frac{K_{1,a}}{K_{1,b}} < 1$. Here, we are confronted with an unusual status in which as soon as we turn on κ -spacetime deformation, the nature of two-body collision alters from elastic to inelastic one. It is important to stress that the overall behavior of the curves in figure 2 are independent of the values of the dimensionless relativistic parameter γ .

⁴In the framework of Newtonian kinematics and for the case of two equal masses $m_1 = m_2$, the ratio $\frac{K_{1,a}}{K_{1,b}}$ in terms of ψ results in the simple relation $\frac{K_{1,a}}{K_{1,b}} = \cos^2 \psi$ which the laboratory scattering angle ψ is limited to the interval $0 \leq \psi \leq \frac{\pi}{2}$. For more details one can see “Classical Mechanics” text book such as Ref. [86]. If $\psi = 0$ then $\frac{K_{1,a}}{K_{1,b}} = 1$, namely, no collision takes place since the speed remains constant before and after the collision and no momentum is transferred to particle-2. However, if $\psi \rightarrow \frac{\pi}{2}$ then $K_{1,a} \rightarrow 0$ ($K_{2,a} = K_{1,b}$) and particle-2 is forwardly scattered with the same kinetic energy before owned by particle-1.

At this point a question arises whether it is possible to remove these unusual phenomenological effects via κ deformation of the momentum conservation law (16). To check this situation, inspired by [49] we modify Eq. (16) as follows

$$\frac{p'_1}{1 - \frac{\alpha}{\kappa} E'_1} = \frac{p'_2}{1 - \frac{\alpha}{\kappa} E'_2} . \quad (45)$$

Let's focus carefully on this modified momentum conservation law in the CM system. Since in the adopted DSR proposal (the MS proposal) one has the spacetime translational invariance, the energy and momentum are conserved. However, DSR proposals are non-linear in essence. This means that 4-momentum of a system of two particles is not just a linear summation of two particles momentums. As a consequence, in passing from SR to DSR, i.e. from linear to non-linear relativity, the additivity of energy and momentum is replaced with non-additivity character. For more insight on this issue we refer the reader to [49]. Now the modified relation (45) can be rewritten as follows

$$p'_1 = \left(1 + \frac{\alpha m^2 c^4}{\kappa^2} (\xi'_2 - \xi'_1) - \left(\frac{\alpha m^2 c^4}{\kappa^2} \right)^2 \xi'_1 \xi'_2 \right) p'_2 , \quad (46)$$

where $m_1 = m_2 = m$. Up to the first order of the modification (i.e. κ^{-1}), the second and third terms on the right hand side of this relation can be neglected. This means that, up to the first order correction, momentum conservation law (16) is still valid and consequently the obtained unusual outcomes are direct consequences arising from extended phenomenological framework at hand.

As the final remark in this section, we note that the apparent energy non-conservation appeared above has been conjectured in NC manifolds and could be due to a discrete (underlying) time evolution [87]. So, Noether's theorem should be correspondingly reformulated for the case of NC spacetime translations.

3 κ -deformed rapidity

In this section we firstly present an important kinematical variable, the rapidity, that relates the momentum of the particle to the dynamics of a heavy-ion reaction. There is the possibility of creation of several particles after collision. The momentum of each particle can be decomposed into a longitudinal component p_l and a transverse component p_t with reference to the collision axis. The longitudinal momentum of a particle, due to its dependence on the velocity of the CM frame with respect to the laboratory frame, is not so conventional variable in literature. Besides, in order to analyze some experimental outputs, it is essential to be able to view them from the CM frame's perspective. As has been mentioned previously, we use the word "CM frame" for center of momentum frame in this context. By introducing a kinematical variable as the rapidity y , one is able simply to choose or change the reference frame. This is arising from the fact that unlike the velocity, the variable y is defined in special relativity to be additive under a sequence of the Lorentz transformations along the same direction. In what follows, we answer the question whether the variable y still remains additive under successive κ -deformed Lorentz transformations (7). In three dimensional space, modified on shell condition (4), can be rewritten as follows

$$E^2 = p_l^2 + M_t^2 , \quad (47)$$

so that

$$M_t^2 = p_t^2 + m^2 \left(1 - \frac{E}{\kappa} \right)^2 , \quad (48)$$

is known as transverse mass. By writing Eq. (47) as $(\frac{E}{M_t})^2 - (\frac{p_l}{M_t})^2 = 1$, then the variable y can be defined in terms of the energy and momentum as follows

$$E = M_t \cosh y, \quad p_l = M_t \sinh y . \quad (49)$$

Therefore, for a κ -deformed special relativity scenario with dispersion relation as Eq. (4), one finds that the rapidity/energy-momentum relation reads as Eq. (49) which is similar to its relativistic counterpart. So, the relation between velocity and rapidity now is obtained from Eq. (49) as

$$u_l \equiv \frac{cp_l}{E} = c \tanh y. \quad (50)$$

This relation gives the rapidity y as follows

$$y = \frac{1}{2} \ln \left(\frac{1 + u_l}{1 - u_l} \right) = \frac{1}{2} \ln \left(\frac{E + p_l}{M_t} \right), \quad (51)$$

since

$$\tanh^{-1} z = \frac{1}{2} \ln \left(\frac{1 + z}{1 - z} \right). \quad (52)$$

To investigate the additivity of rapidity under a sequence of the κ -deformed Lorentz transformations, we consider the transformation of the momentum vector under a change of the reference frame along the collision axis. It is obvious that under such a transformations, p_t and M_t are unchanged. Therefore, under a κ -deformed Lorentz transformation, the energy and longitudinal component of the momentum transform as

$$E' = \frac{\gamma_c}{\alpha} \left(E + \frac{u_c}{c^2} p_l \right), \quad p'_l = \frac{\gamma_c}{\alpha} \left(p_l + \frac{u_c}{c^2} E \right). \quad (53)$$

Note that here a prime marks the quantities that are measured by an observer in the laboratory system which moves with the velocity u_c with respect to the CM frame of reference in which the energy E and momentum p_l are measured. Putting Eq. (47) along with $\gamma_c = \cosh y_c$ and $\gamma_c u_c = \sinh y_c$ derived from Eq. (50) into Eq. (53), we get

$$E' = \frac{M_t}{\alpha} \cosh(y + y_c), \quad p'_l = \frac{M_t}{\alpha} \sinh(y + y_c). \quad (54)$$

Therefore, the rapidity y' as seen in the laboratory frame can be read as follows

$$y' = \cosh^{-1} \left(\frac{\cosh(y + y_c)}{\alpha} \right). \quad (55)$$

While this relation reflects the fact that rapidity is not an additive quantity under a sequence of the κ -deformed Lorentz transformations, for the case $\alpha = 1$ (that is, under standard Lorentz transformations) this relation restores the additivity property, i.e, $y' = y + y_c$. Therefore, in passing from the standard SR to κ -deformed SR, one loses the additivity nature of rapidity. This is much similar to losing additivity rule of speeds in passing from Galilean relativity to special relativity. Note that relation (55) is a direct result of the nonlinear representation of Lorentz transformations. Technically, as is shown in Ref. [76], the Magueijo-Smolín proposal of DSR can be realized via deformed translation invariance without need to deformation of the LS (this is also the case for all other proposals of DSR, see [74]). The κ -Minkowski spacetime modified translation generator in the MS model of DSR is given by $t^\mu = \frac{p^\mu}{1 - \alpha p/\kappa}$. In this respect, the origin of deviations from SR observed in this paper comes back to the deformed translation invariance since this really controls how momentum of particles behaves in collisions.

4 Conclusion

Noncommutative geometry (spacetime) is undoubtedly one of the richest framework among other alternatives to pursue physics at the Planck scale. In this paper, we were concerned on a particular form of noncommutative spacetimes which highly takes care of Lorentz Symmetry and known as κ -Minkowski spacetime. The noncommutative parameter κ can be conceived as quantum gravity scale

where high energy physics is trying to unravel its mysteries. The mentioned noncommutative parameter is observer independent which leads to a natural connection between κ -Minkowski spacetime and Doubly Special Relativity theories (DSRs). Therefore, noncommutative geometry parameter κ is responsible for cutoff energy scale featured in DSR theories [48, 49] as the second invariant. In the new framework, we are faced with DSR formalism as a nonlinear κ extension of the Lorentz transformations and dispersion relations of Special Relativity. In this work, through focusing on the Magueijo-Smolín version of DSR theories [49], we have investigated how such energy upper bound will affect some relativistic kinematical parameters in a typical collision of particles. As a first step, by applying κ -deformed Lorentz transformations (6), (7) along with modified mass shell condition (4), we have investigated the effect of the Planck energy cutoff on the elastic scattering processes for the case that no new particles are produced in the process. We have shown (as figure 1 indicates) that the existence of α -term caused by the κ -Minkowski spacetime, leads increment of maximum included scattering angle ϕ_{Max} relative to the standard case. While the classical and special relativistic kinematics indicate that the maximum included scattering angle is bounded as $\phi_{Max} \leq \frac{\pi}{2}$, the κ -deformed nonlinear extension of special relativity violates this bound since for angles larger than $\frac{\pi}{2}$, there is probability of detection of scattered particles. In the same way, by assuming that the recoil angle η is negligible, we have treated the effect of the Planck energy upper bound (κ modification) on the kinetic energy K of the scattered particle. Surprisingly, in contrast to classical mechanics and even special theory of relativity, we have observed that for the case $\psi = 0$ and $\alpha > 1$, $\frac{K_{1,a}}{K_{1,b}} < 1$ as figure 2 shows. Considering the fact that for the case $\alpha = 1$ one recovers the standard result $\frac{K_{1,a}}{K_{1,b}} = 1$, in the context of the κ deformed extension of the special relativity one tempts to think that spacetime seems to have some sort of dissipative effects at quantum gravity scale. The apparent energy non-conservation appeared in our context could be due to a discrete (underlying) time evolution as shown in Ref. [87]. It is also expected that the Noether's theorem must be correspondingly reformulated in the present case of NC spacetime translations. As an important result, we have shown that unlike the special relativity, rapidity in κ -Minkowski spacetime is no longer an additive quantity under a sequence of κ -deformed Lorentz transformations. This is reminiscent of the fact that one loses the additive nature of speeds in passing from Galilean relativity to special relativity. We note that the observed deviations from SR in this paper originate from the deformed translation invariance, not necessarily to the deformed LS. In fact, this deformed translation invariance is that controls how momentum of particles behaves in collisions. The outputs of our study support the idea that the main feature of DSR theories is the deformed translation symmetry, not the deformed LS as has been discussed also in [76, 74].

Finally, about the algebraic structure of κ -Minkowski spacetime if we look at the full algebra generated by boosts, rotations and translations, then the algebra is really the Lorentz algebra. Depending on the choice of realization, even the translation part can be un-deformed and therefore having the full Poincaré algebra. But this is only on the algebra level, that is what governs the one particle representations. However the coalgebraic sector is deformed, leading to nontrivial multi-particle states, which are the ones appearing in collisions. The κ -Minkowski algebra is compatible with deformed Poincaré algebra, usually κ -Poincaré algebra (but one can also go beyond this in general to deformations of igl-Hopf algebra) for which we have that the Lorentz sector is intact, but depending on realization the translation part is deformed. Only in the classical basis (or natural realization) the translation part is also intact. Anyway, for any realization (natural, bicrossproduct etc) the coalgebraic sector is deformed, meaning that the coproduct of boosts, rotation and translation is not primitive, and this gives some new interesting phenomenon for the multiparticle states, and therefore collisions [88].

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